Lecture 20

<u>Proposition</u>: Suppose $\beta = \{\vec{v}_1, \dots, \vec{v}_K\}$ is a basis for a subspace $W \subset \mathbb{R}^n$. Then $\forall\, \vec{w} \in W$ $\exists$ unique scalers $x_1, \dots, x_K \in \mathbb{R}$ such that

$$\vec{w} = x_1\vec{v}_1 + \dots + x_K\vec{v}_K$$

<u>Proof</u> - Since β is a basis for W, we have $W = \text{span}\,\beta$. So given $\vec{w} \in W$ I have

$\vec{w} = x_1\vec{v}_1 + \dots + x_K\vec{v}_K$ for some scalers $x_1 \dots x_K \in \mathbb{R}$.

Suppose also: $\vec{w} = y_1\vec{v}_1 + \dots + y_K\vec{v}_K$ for some $y_1, \dots, y_K \in \mathbb{R}$.

$$\vec{0} = (x_1 - y_1)\vec{v}_1 + \dots + (x_K - y_K)\vec{v}_K$$

$\Longrightarrow x_1 - y_1 = \dots = x_K - y_K = 0$, because β is linearly independent.

i.e. $x_1 = y_1 \ \dots \ x_K = y_K$

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• We call $x_1, \dots, x_K$ the <u>coordinates</u> of $\vec{w}$ relative to β and write

$[\vec{w}]_\beta = \begin{pmatrix} x_1 \\ \vdots \\ x_K \end{pmatrix}$ = "coordinate vec" relative to β.

$\boxed{\text{Example}}$

$$W = \text{span}\underbrace{\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 8 \\ -7 \\ 2 \end{bmatrix} \right\}}_{\beta}$$

If $\vec{w} \in W$ has $[\vec{w}]_\beta = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$\vec{w} = 1 \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 6 \\ 3 \end{bmatrix} + 3 \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \\ 8 \\ -7 \\ 2 \end{bmatrix}$$

Rmk - Let $S_\beta = [\vec{v}_1 \ldots \vec{v}_K]$. Then $[\vec{w}]_\beta = \begin{pmatrix} x_1 \\ \vdots \\ x_K \end{pmatrix}$ means

$$\vec{w} = x_1\vec{v}_1 + \ldots + x_K\vec{v}_K = S_\beta [\vec{w}]_\beta$$

$$\vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \in \mathbb{R}^n \quad \text{and} \quad \mathcal{E} = \{\vec{e}_1, \ldots, \vec{e}_n\} \subset \mathbb{R}^n$$

Then $[\vec{w}]_{\mathcal{E}} = \vec{w}$ = "standard coordinates" for $\vec{w}$.

$\boxed{\text{Example}}$

$$\beta = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \subset \mathbb{R}^2$$

Let $\vec{w} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$. Then $[w]_\beta = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

because $\begin{bmatrix} 7 \\ 5 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

What if $\lambda = \left\{ \begin{bmatrix} 17 \\ 31 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \end{bmatrix} \right\}$

What is $[\vec{w}]_\lambda$?

ANS 1 - Need x_1, x_2 such that

$$\begin{bmatrix} 7 \\ 5 \end{bmatrix} = x_1 \begin{bmatrix} 17 \\ 31 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 7 \end{bmatrix}$$

i.e. I need to solve $\left[\begin{array}{cc|c} 17 & -3 & 7 \\ 31 & 7 & 5 \end{array}\right]$

ANS 2 - We know $\vec{w} = \lambda_\beta [\vec{w}]_\beta \rightarrow \lambda_\beta^{-1} \vec{w} = [\vec{w}]_\beta$

So:

$$\begin{bmatrix} 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 17 & -3 \\ 31 & 7 \end{bmatrix} [\vec{w}]_\beta$$

So: $$\frac{1}{17(7)-(-3)(31)} \begin{bmatrix} 7 & 3 \\ -31 & 17 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \end{bmatrix} = [\vec{w}]_\beta$$

$\boxed{\text{Ex}}$ Suppose linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ given by $T(\vec{v})$ = orthogonal projection of $\vec{v}$ onto the plane $P = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : 2x_1 + x_2 - x_3 = 0 \right\}$

What is $[T]$?

$$[T] = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \end{bmatrix}$$

Need another approach!

Idea: Pick a more convinient basis.

<u>Observations</u> - If $\vec{v}_3$ is orthogonal to P then $T(\vec{v}_3) = 0$.

$$\vec{v}_3 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

- If $\vec{v}_1, \vec{v}_2 \in P$ then $T(\vec{v}_1) = \vec{v}_1$, $T(\vec{v}_2) = \vec{v}_2$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

So take $\beta = \{ \vec{v}_1 , \vec{v}_2 , \vec{v}_3 \}$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

if $[\vec{w}]_\beta = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ then $\vec{w} = x_1 \vec{v}_1 + x_2 \vec{v}_2 + x_3 \vec{v}_3$

$\rightarrow T(\vec{w}) = x_1 T(\vec{v}_1) + x_2 T(\vec{v}_2) + x_3 T(\vec{v}_3)$

$\rightarrow T(\vec{w}) = x_1 v_1 + x_2 v_2 + x_3 \vec{0}$

so $$[T(\vec{w})]_\beta = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{[T]_\beta} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{= [\vec{w}]_\beta}$$

i.e. $[T(\vec{w})]_\beta = [T]_\beta [\vec{w}]_\beta$

finally $T(\vec{w}) = S_\beta [T(\vec{w})]_\beta$ and $\vec{w} = S_\beta [\vec{w}]$

So $T(\vec{w}) = S_\beta [T(\vec{w})]_\beta = S_\beta [T]_\beta [\vec{w}]_\beta$

$$= \underbrace{\left(S_\beta [T]_\beta S_\beta^{-1} \right)}_{[T]} \vec{w}$$

So $$[T] = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}^{-1}$$

$$[T] = \frac{1}{6} \begin{bmatrix} 2 & -2 & 2 \\ -1 & 5 & 1 \\ 2 & 1 & 5 \end{bmatrix}$$